# Martingales in Banach spaces

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## 1 Three line summary

- Conditional expectations exist in a natural way for simple functions, by taking extensions they also exist for integrable functions to a Banach space $L^1(\Omega \to E)$.
- Using conditional expectations we can define what a martingale is just like in the real case.
- The space of continuous $p$-integrable martingales is a Banach space.

## 2 Why should I care?

Banach valued martingales form the basis of SPDEs. This is because analogously to Itô integration of real-valued processes. Integrating against a Wiener process valued in a Banach space the same will produce a square integrable continuous martingale.

## 3 Conditional expectation

In graduate-level probability courses, given a $\sigma-$algebra $\mathcal{G}$ one shows that by applying Radon-Nikodyn's theorem, for any real-valued random variable $X \in L^1(\Omega \to R)$ there exists a conditional expectation $\mathbb{E}_{\mathcal{G}}[X]$ verifying that

$$\int_A \mathbb{E}_{\mathcal{G}}[X] = \int_A X, \quad \forall A \in \mathcal{G}.$$

Of course, now that we have created an integral for integral random variables to a Banach space $L^1(\Omega \to X)$ we would like to see whether such a conditional expectation also exists for these functions. If we are given a simple function

$$X = \sum_{k=1}^{n} x_k 1_{A_k}, \quad x_k \in E, A_k \in \mathcal{G}.$$

It is a simple calculation to show that, since $1_{A_k}$ are real-valued and thus $\mathbb{E}_{\mathcal{G}}[1_{A_k}]$ are well defined, then

$$\mathbb{E}_{\mathcal{G}}[X] = \sum_{k=1}^{n} x_k \mathbb{E}_{\mathcal{G}}[1_{A_k}],$$

verifies the desired formula. Furthermore, we have that $\mathbb{E}_{\mathcal{F}}$ is a linear, and pointwise continuous operator with

$$\|\mathbb{E}_{\mathcal{G}}[X]\| \leq \sum_{k=1}^{n} \|x_k\| \mathbb{E}_{\mathcal{G}}[1_{A_k}] = \mathbb{E}_{\mathcal{G}}\left[\sum_{k=1}^{n} \|x_k\| 1_{A_k}\right] = \mathbb{E}_{\mathcal{G}}\left[\|X\|\right].$$

This allows us to show the following

**Theorem 1** (Existence and uniqueness of conditional expectation)**.** *Let $X \in L^1(\Omega \to E)$ for some Banach space $E$. Then $X$ has a conditional expectation satisfying*

$$\|\mathbb{E}_{\mathcal{G}}[X]\| \leq \mathbb{E}_{\mathcal{G}}\left[\|X\|\right].$$

*Proof.* We have already proved the above inequality for simple processes. By the previous post, [1] we can take $X_n$ converging to $X$ in $L^1(\Omega \to E)$ to obtain that

$$\begin{aligned} \|\mathbb{E}_{\mathcal{G}}[X_n - X_m]\| &\leq \mathbb{E}_{\mathcal{G}}\left[\|X_n - X_m\|\right] \\ &\implies \mathbb{E}[\|\mathbb{E}_{\mathcal{G}}[X_n] - \mathbb{E}_{\mathcal{G}}[X_m]\|] \leq \mathbb{E}\left[\|X_n - X_m\|\right] \to 0 \end{aligned}$$

As a result, $\mathbb{E}_{\mathcal{G}}[X_n]$ is a Cauchy sequence in $L^1(\Omega \to E)$ and converges to some function $Y$, passing to the limit in the defining equation for the conditional expectation shows that $Z = \mathbb{E}_{\mathcal{G}}[X]$. Finally, to prove uniqueness we have that if both $Z_1, Z_2$ satisfy

$$\int_A Z_1 = \int_A X = \int_A Z_2, \quad \forall A \in \mathcal{G}.$$

Then using the linearity of the integral we obtain that $w(Z_1) = w(Z_2)$ for all linear function $w$, so $Z_1 = Z_2$. □

# 4 Martingales

Okay, so we leveraged some inequalities to prove the existence of a conditional expectation. This done, the following definition mimicking the real case is quite natural

**Definition 1.** *Let $\{M(t)\}_{t\in I}$, be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in I}$. The process $M$ is called an $\mathcal{F}_t$-martingale, if:*

1. *$M(t) \in L^1(\Omega \to E)$ for all $t \in I$*
2. *$M(t) : \mathcal{F}_t \to \mathcal{B}(E)$ for all $t \in I$,*
3. *$\mathbb{E}_{\mathcal{F}_s}[M(t)] = M(s)$ for all $s \leq t$.*

The concept of submartingale is defined by replacing the equality in 3. with a $\geq$. Let us abbreviate $\mathbb{E}_{\mathcal{F}_t}$ by $\mathbb{E}_t$. Then, as in the real case, we have the following.

**Lemma 1** (Norm is submartingale)**.** *Let $M(t)$ be a martingale, then $\|M(t)\|$ is a martingale*

*Proof.* We recall that, by the Hahn Banach theorem, it holds for any metric space that given $y \in E$

$$\|z\| = \sup_{\ell \in E^* : \|\ell\| = 1} \ell(z)$$

As a result, by the linearity of the integral and abbreviating the supremum to just $\sup_\ell$,

$$\begin{aligned} \|M(s)\| = \|\mathbb{E}_s[M(t)]\| = \sup_\ell \ell\left(\mathbb{E}_s[M(t)]\right) = \sup_\ell \|\mathbb{E}_s\left[\ell(M(t))\right]\| \\ \leq \mathbb{E}_s\left[\sup_\ell \ell(M(t))\right] = \mathbb{E}_s\left[\|M(t)\|\right] \end{aligned}$$

□

Let us recall the following result for real-valued martingales

**Lemma 2** (Doob's maximal Martingale inequality)**.** *Let $\{X_k\}_{k=1}^\infty$ be a real-valued sub-martingale. Then it holds that*

$$\left\| \max_{k\in\{1,\dots,n\}} X_k \right\|_{L^p(\Omega)} \leq \frac{p}{p-1} \|X_n\|_{L^p(\Omega)}$$

*As a consequence, if $X_t, t \in [0,T]$ is left (or right) continuous then*

$$\left\| \max_{t\in[0,T]} X_k \right\|_{L^p(\Omega)} \leq \frac{p}{p-1} \|X_T\|_{L^p(\Omega)}.$$

The idea of the above result is that, since $X_k$ is a submartingale, $X_k \lesssim X_{k+1} \lesssim ... \lesssim X_n$. Getting from the continuous to the discrete case is possible by using the continuity of $X$ and approximating it on some finer and finer mesh $t_0, ..., t_n$. This said, applying Doob's maximal martingale inequality together with the Lemma 1 gives that

**Theorem 2** (Maximal Inequality)**.** *Let $p > 1$ and let $E$ be a separable Banach space. If $M(t)$, is a right-continuous $E$-valued $\mathcal{F}_t$-martingale, then*

$$\left(E\left(\sup_{t\in[0,T]} \|M(t)\|^p\right)\right)^{\frac{1}{p}} \leq \frac{p}{p-1} \sup_{t\in[0,T]} \left(E\left(\|M(t)\|^p\right)\right)^{\frac{1}{p}}$$
$$= \frac{p}{p-1} \left(E\left(\|M(T)\|^p\right)\right)^{\frac{1}{p}}$$

*Proof.* This follows by using that $\|M(t)\|$ is a sub-martingale and Doob's maximal inequality. □

Doob's inequality is essentially an equality between different function norms we can place on the space of continuous Martingales and will provide a very powerful tool later on.

**Corollary 1.** *Let $M$ be a (left or right) continuous martingale to a separable Banach space $E$. Then the following are equivalent*

- $M \in \hat{L}^\infty([0,T] \to \hat{L}^2(\Omega \to E))$
- $M \in \hat{L}^2(\Omega \to \hat{L}^\infty([0,T] \to E))$
- $\mathbb{E}[\|M(T)\|^2] < \infty$

*Where we recall from the previous post that $\hat{L}^p$ symbolizes that $M$ may not be separately valued and only have an integrable norm. That said, the same reasoning shows that the above result also holds for the integrable $L^p$ spaces.*

A useful space of Martingales is as follows

**Definition 2.** *Let* $M(t)$ *be a* $E$ *valued martingale with index set* $I = [0, T]$, *then we define*

$$\mathcal{M}_T^2(E) := \left\{ \textit{continuous martingales } M : \mathbb{E}[\|M(T)\|^2] < \infty \right\}$$

*and give it the norm*

$$\|M\|_{\mathcal{M}_T^2(E)} := \mathbb{E}[\|M(T)\|^2].$$

By Theorem 2 we have that

$$\mathcal{M}_T^2(E) \subset \hat{L}^\infty([0,T] \to \hat{L}^2(\Omega \to E)) \cap \hat{L}^2(\Omega \to \hat{L}^\infty([0,T] \to E)).$$

and that any of the norms of these spaces is equivalent to the one set on $\mathcal{M}_T^2(E)$. This is useful in the following result

**Proposition 1.** *Let* $E$ *be a separable Banach space, then* $\mathcal{M}_T^2(E)$ *is a Banach space.*

*Proof.* By the previous observation and the completeness of the $\hat{L}^p$ spaces proved in the previous post, $\mathcal{M}_T^2(E)$ is a subspace of a Hilbert space. As a result, it is sufficient to show that it is closed. Let $M_n$ converge to $M$. Then, by the equivalence of the norms we have that $M_n(t) \to M(t) \in \hat{L}^1(\Omega \to E) \subset \hat{L}^2(\Omega \to E)$ so that for all $A \in \mathcal{F}_s$

$$\int_A M(s) d\mathbb{P} = \lim_{n\to\infty} \int_A M_n(s) d\mathbb{P} = \lim_{n\to\infty} \int_A M_n(t) d\mathbb{P} = \int_A M(t) d\mathbb{P}.$$

This shows that $M$ is a martingale. Furthermore, as was seen in the previous post, there exists a subsequence $M_{n_k}$ such that

$$\lim_{n\to\infty} M_{n_k}(\cdot, \omega) = M(\cdot, \omega) \in \hat{L}^\infty([0,T] \to E) \quad \textit{a.e.} \quad \omega \in \Omega$$

Since $M_{n_k}(\cdot, \omega)$ are continuous and continuity is preserved by uniform limits this proves that $M$ is continuous almost everywhere. This concludes the proof. □

In future installments, we will prove that a Banach valued Wiener process belongs to this space and use it to define the stochastic integral that leads to the construction of SPDEs.

**Proposition 2.** *Let $W(t)$ be a $E$ valued $\Sigma$-Wiener process with respect to a filtration $\mathcal{F}_t$. Then $W(t) \in \mathcal{M}_T^2(E)$.*

*Proof.* It is a martingale as it is adapted and, given $A \in \mathcal{G}_S$ and $u \in E$, by the linearity of the integral and Independence of $W(t) - W(s)$ with $\mathcal{G}_s$

$$\begin{aligned}\left\langle \int_A W(t) - W(s) d\mathbb{P}, u \right\rangle &= \int_A \langle W(t) - W(s), u \rangle \, d\mathbb{P} \\ &= \mathbb{P}(A)\mathbb{E}[\langle W(t) - W(s), u \rangle] = 0\end{aligned}$$

As a result

$$\int_A W(t) d\mathbb{P} = \int_A W(s) d\mathbb{P} = 0 \quad \forall A \in \mathcal{G}_s \implies \mathbb{E}_s[W(t)] = W(s).$$

Finally, we have that $\mathbb{E}[W(t)^2] = t < \infty$ for all $t$ and $W$ is continuous by construction. This concludes the proof. $\square$

# References

[1] L. Llamazares, The bochner integral (2022).
URL `https://liamllamazares.github.io/2022-05-27-The-Bochner-integral/`